

A partition Π of a set S is a subdivision of S into disjoint, nonempty subsets

Ex $S = \{\text{apple}, \text{bird}, 1, 2, 3\}$

ex of partition

$$S = \{\text{apple}, 1\} \cup \{\text{bird}, 2\} \cup \{3\}$$

Ex $S = \mathbb{Z}$.

$$\mathbb{Z} = \text{Even} \cup \text{Odd}$$

$$\text{Even} = \{2n : n \in \mathbb{Z}\}$$

$$\text{Odd} = \{2n+1 : n \in \mathbb{Z}\}$$

Equivalence Relation

S a set. An equivalence relⁿ on S is a relⁿ that holds between pairs of elts of S , written $a \sim b$ if $a, b \in S$, that's required to satisfy the following:

transitivity if $a \sim b$ and $b \sim c$ then $a \sim c$

symmetry if $a \sim b$ then $b \sim a$

reflexivity $a \sim a$.

$$\forall a, b, c \in S.$$

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Ex $S = \mathbb{Z}$. Define relⁿ $a \sim b$ if the prime numbers that divide a are the same as the prime #'s that divide b .

Ex p, q distinct primes
 $p \sim p^2 \sim p^3$
 $p \not\sim pq \sim p^2 q^5$

Ex Conjugacy in a gp. G a gp.

Define $a \sim b$ if $\exists g \in G$ s.t. $b = gag^{-1}$.

transitivity: $a \sim b$ and $b \sim c$

$$\Downarrow$$

$$\exists g_1 \in G \text{ s.t. } b = g_1 a g_1^{-1}$$

$$\exists g_2 \in G \text{ s.t. } c = g_2 b g_2^{-1}$$

$$\begin{aligned} c &= g_2 b g_2^{-1} = g_2 (g_1 a g_1^{-1}) g_2^{-1} \\ &= (g_2 g_1) (a) (g_2 g_1)^{-1} \end{aligned}$$

$$\Rightarrow c \sim a.$$

symmetry: $a \sim b \Rightarrow \exists g \in G \text{ s.t. } b = gag^{-1}$

$$\begin{aligned} \Rightarrow a &= g^{-1} b g \\ &= g^{-1} b (g^{-1})^{-1} \Rightarrow b \sim a \end{aligned}$$

reflexive $a \sim a$: $a = 1 \cdot a \cdot 1^{-1}$.

Ex Cosets G a gp, $H \leq G$ a subgp

Define $a \sim b$ if $\exists h \in H$ s.t. $b = ah$

This is an equiv relⁿ

transitivity: $a \sim b$ and $b \sim c$ then

$$\begin{array}{ll} b = ah_1 & c = bh_2 \\ \text{for some } h_1 \in H & \text{for some } h_2 \in H \end{array}$$

$$\Rightarrow c = bh_2 = (ah_1)h_2 = a(h_1h_2)$$

$\hat{H} \text{ b/c } H \text{ is a subgp.}$

Symmetry $a \sim b$: $b = ah$ for some $h \in H$

$\Rightarrow a = bh^{-1}$. But, $h^{-1} \in H$ b/c H is closed under inverses

$$\Rightarrow b \sim a.$$

reflexivity $a \sim a$ b/c $a = a \cdot 1$, $1 \in H$.

Suppose S is a set w/ an equiv relⁿ, and $a \in S$.

Then the equivalence class of a is

$$S_a = \{s \in S: a \sim s\}$$

Non-example $S = \mathbb{R}$. Define $a \sim b$ if $a \leq b$.

transitive: $a \leq b$ and $b \leq c \Rightarrow a \leq c$

transitive: $a \leq b$ and $b \leq c \Rightarrow a \leq c$

reflexive: $a \leq a$.

not symm: $a \leq b \not\Rightarrow b \leq a$.

Prop: An equivalence relⁿ on a set S determines a partition of S , via the equivalence classes. Conversely, a partition of S determines an equivalence relⁿ.

Pf: Suppose S is a disjoint union of nonempty subsets S_α
$$S = \bigcup_{\alpha \in A} S_\alpha$$

"disjoint union"

Define $a \sim b$, a relⁿ on S , if a and b are both in S_α for some subset S_α of the partition

This is an equivalence relⁿ (check this)

Conversely, suppose given an equivalence relⁿ $\sim$ on S .

Let $S_a = \{s \in S : a \sim s\}$ be the equiv class of a .

Claim: The equiv. classes partition S :

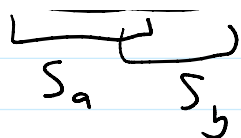
$$S = \bigcup_a S_a$$
 is a partition

I.e.

1) • every elt of S is in some S_a

2) • If $S_a \cap S_b \neq \emptyset$, then $S_a = S_b$





1) is clear, b/c if $a \in S$ then $a \in S_a$. This is because $a \sim a$ by reflexivity.

2) Suppose $c \in S_a \cap S_b$. Then $a \sim c$ and $b \sim c$.

$$\Rightarrow a \sim c \text{ and } c \sim b \Rightarrow a \sim b$$

(by symmetry) (by transitivity)

WTP: $S_a = S_b$.

Suppose $x \in S_b$. Then $a \sim b$, $b \sim x \Rightarrow a \sim x$
 $\Rightarrow x \in S_a$.

$$\Rightarrow S_b \subseteq S_a. \Rightarrow S_a = S_b.$$

Similarly, $S_a \subseteq S_b$

□

Ex Fix $n \in \mathbb{Z}$, $n \neq 0$. Define $a \sim b$ if $a - b \in \mathbb{Z}n$.
 This is an equivalence rel^n , b/c it is a special case
 of cosets for $G = (\mathbb{Z}, +)$, $H = \mathbb{Z}n$ the subgroup.

Ex Suppose $n = 10$. Then $a \sim b$ if a and b have the
 same final digit (if expressed in base 10)

Equivalence classes

$$\underbrace{10\mathbb{Z}, 10\mathbb{Z} + 1, 10\mathbb{Z} + 2, \dots, 10\mathbb{Z} + 9}$$

$\{0 \in \mathbb{Z}, 1 \in \mathbb{Z}^+, 1 \in \mathbb{Z}^+ a, \dots, 1 \in \mathbb{Z}^+ 1\}$
(0 equiv. classes

Ex S, T sets, $f: S \rightarrow T$

Then define $a \sim b$ if $f(a) = f(b)$.
This is an equivalence rel.

Equiv. classes If $t \in T$, define

$$f^{-1}(t) = \{s \in S : f(s) = t\}$$

Called "the fibers" of the map f .

The equiv. classes are the non-empty fibers; they partition S .
